Econ 504
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Econ 504
Second Midterm
Answers

1. We get the same solution regardless of whether we use average utility or the present value of utility.

   (a) Using average utility, if the catastrophe happens in current period, the value function is $V_C = -100$
   If the catastrophe does not happen in current period, denote the value function by V. There are 2 choices in this case.
   If the investment is made,

   $$V_I = (1-\delta)\,5 + \delta\,[0.1V_C + 0.9V]\,.$$

   If the investment is not made,

   $$V_N = (1-\delta)\,10 + \delta\,[0.2V_C + 0.8V]\,.$$

   $V = \max\{V_I, V_N\}$. We want to find the range of δ so that $V = V_I$. Making the appropriate substitutions, we want to find δ such that

   $$\begin{gathered} V_I = (1-\delta)\,5 + \delta\,[-10 + 0.9V_I] \\ V_N = (1-\delta)\,10 + \delta\,[-20 + 0.8V_I] \\ V_I \geq V_N \\ 0 \leq \delta < 1 \end{gathered}$$

   The first expression yields

   $$V_I = \frac{5 - 15\delta}{1 - 0.9\delta}$$

   Thus, the condition $V_I \geq V_N$ becomes

   $$\frac{5 - 15\delta}{1 - 0.9\delta} \geq (1-\delta)10 + \delta\left[-20 + .8\frac{5 - 15\delta}{1 - 0.9\delta}\right]$$

   Manipulating this yields that investment is optimal if $\frac{1}{3} \leq \delta < 1$.

(b) If instead we use the present value of utility, the problem becomes to find δ such that

$$\begin{gathered} V_I' = 5 + \delta\left[-\tfrac{10}{1-\delta} + 0.9V_I'\right] \\ V_N' = 10 + \delta\left[-\tfrac{20}{1-\delta} + 0.8V_I'\right] \\ V_I' \geq V_N' \\ 0 \leq \delta < 1 \end{gathered}$$

The required δ will be the same.

2. (a) By the revelation principle, it suffices to consider truth telling equilibria of revelation auctions. In any such NE, if a player has value v_i but reports v_i^* then his expected payoff is

$$Q_i(v_i^*)v_i - T_i(v_i^*).$$

Since it is optimal in the truth telling equilibrium to report the true value,

$$\begin{aligned} U_i(v_i) = Q_i(v_i)v_i - T_i(v_i) &\geq Q_i(v_i^*)v_i - T_i(v_i^*) \\ U_i(v_i^*) = Q_i(v_i^*)v_i^* - T_i(v_i^*) &\geq Q_i(v_i)v_i^* - T_i(v_i) \end{aligned}$$

for any $v_i, v_i^* \in [0, 100]$. Without loss of generality, assume $v_i^* > v_i$. Then

$$\begin{aligned} U_i(v_i^*) - U_i(v_i) &\geq Q_i(v_i)v_i^* - T_i(v_i) - [Q_i(v_i)v_i - T_i(v_i)] \\ &= Q_i(v_i)[v_i^* - v_i] \end{aligned}$$

$$\begin{aligned} U_i(v_i^*) - U_i(v_i) &\leq Q_i(v_i^*)v_i^* - T_i(v_i^*) - [Q_i(v_i^*)v_i - T_i(v_i^*)] \\ &= Q_i(v_i^*)[v_i^* - v_i] \end{aligned}$$

Thus

$$Q_i(v_i) \leq \frac{U_i(v_i^*) - U_i(v_i)}{v_i^* - v_i} \leq Q_i(v_i^*)$$

Since this holds for any v_i, v_i^*, it follows that

$$U_i(v_i) = U_i(0) + \int_0^{v_i} Q_i(x)\, dx.$$

(b) Substituting the definition of $U_i(v_i)$ into the integral expression above,

$$Q_i(v_i)v_i - T_i(v_i) = U_i(0) + \int_0^{v_i} Q_i(x)\, dx,$$

hence

$$T_i(v_i) = -U_i(0) + Q_i(v_i)v_i - \int_0^{v_i} Q_i(x)\, dx.$$

This implies that the total expected transfer is

$$\begin{aligned}\mathbb{E}_v\left[\sum_i t_i\right] &= \mathbb{E}_v\left[\sum_i T_i(v_i)\right]\\ &= -\sum_i U_i(0) + \mathbb{E}_v\left[\sum_i\left[Q_i(v_i)v_i - \int_0^{v_i} Q_i(x)\,dx\right]\right]\end{aligned}$$

Efficiency implies that for each i, $Q_i(v_i) = \mathbb{P}[v_i \geq v_j\ \forall j]$. Thus, the only variables that the seller controls are the values of the $U_i(0)$. From the integral expression, IR holds iff $U_i(0) \geq 0$. Thus, the total expected transfer, assuming efficiency and IR, is maximized by setting $U_i(0) = 0$ for all i. But the truthtelling equilibrium of the second price auction is efficient and has $U_i(0) = 0$, and thus the truthtelling equilibrium of the second price auction maximizes expected revenue across all efficient NE of all auctions.

3. (a) In any separating equilibrium, entry occurs after the high cost type executes his equilibrium strategy. Consider, then, any strategy profile under which (a) entry occurs whenever the high cost type executes his strategy and (b) the high cost type does not play 3 for certain in the first period. This cannot be a separating equilibrium profile, because the high cost type can increase profits by playing 3 for certain in the first period: switching to 3 for certain in the first period increases first period profits and cannot increase the probability of entry following the action of the high cost firm (since this probability is already 1). It follows that in any separating equilibrium, the high cost type plays 3 for certain in the first period.

   (b) In any separating equilibrium, the entrant stays out if she sees the output associated with the low cost firm. Consider any strategy profile in which the entrant stays out if she sees 5 in the first period. Then this cannot be a separating equilibrium profile because the high cost firm can increase profits by deviating and producing 5 instead of 3 in the first period (contradicting (a)), earning $5 + 9 = 14 > 9 + 1 = 10$.

   (c) In one separating equilibrium, the low cost firm produces 6 in the first period. The entrant assigns probability 1 to the incumbent being low cost if it observes 6, 0 otherwise. (Other probabilities will work; these are simply the easiest to use.) The entrant stays out if it sees 6, enters otherwise.

   To check sequential rationality, note that if the incumbent is low cost, the best possible deviation for it is to 5, which earns 25 (vs. 24) but only 13 next period (because of entry) for a total of 38 (vs. 49). If the incumbent is high cost, the only deviation that makes any sense at all (given that

3 already maximizes per period profit) is to 6 (which deters entry). But then the high cost firm earns $0 + 9 = 9$ which is less than the $9 + 1 = 10$ earned if the firm sticks to the stated strategy.

Bayes consistency is immediate. As for consistency, just take any strategy sequence in which the probability of the low cost incumbent deviating to anything other than 6 is ε^2 while the probability of high cost incumbent doing so is ε.

There is another separating sequential equilibrium in which the incumbent produce 7, rather than 6, if low cost. The entrant believes the firm is low cost if it observes 7. The entrant believes the incumbent is high cost if it sees any other output. The entrant stays out if it sees 7, enters otherwise. This is a bad equilibrium in the sense that to support it, the entrant must believe that an output of 6 in the first period indicates a high cost incumbent. This is, arguably, implausible, because the high cost incumbent has no incentive to deviate from 3 to 6, even if by doing so it could trick the entrant into believing that its cost was actually low. In contrast, the low cost incumbent has an incentive to deviate from 7 to 6 provided the entrant believed, correctly, that it was low cost. For the reason just sketched, the separating equilibrium in which the incumbent produces 7 violates a condition called the intuitive criterion.

(d) There is a PBE pooling equilibrium in which both types of incumbent produce 5 units in the first period. If the entrant observes 5, it assigns probability 1/2 to the incumbent being low cost. If the entrant observes any other output, the probability is 0. The entrant stays out if it sees 5. Otherwise, the entrant enters.

Again, it is easy to check sequential rationality. In particular, an entrant should stay out if it sees a 5 since in expectation it earns $-6/2+4/2 = -1$. And, again, it is easy to check consistency.

There is another pooling sequential equilibrium in which both types of the incumbent produce 4 units in the first period. This is a bad equilibrium in the sense that it violates the intuitive criterion. The reason, much as in the bad equilibrium in (c), is that the equilibrium is supported by having the entrant believe that if it sees an output of 6 then the incumbent is high cost, even though only a low cost incumbent has a possible incentive to deviate to 6.