Econ 504
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Econ 504
Final - Answers

1. *Counting equilibria.* The equilibria are all symmetric and of the form, if there are $K \leq L$ strategies that get positive probability then each of these strategies gets probability $1/K$. Thus the number of equilibria equals the number of non-empty subsets of L, which is $2^L - 1$.[1]

   Remark. Thus, the set of all equilibria can be exponential in the number of strategies. The example is robust; small changes in the payoffs result in small changes in the equilibria but we still have a total of $2^L - 1$ equilibria.

2. *Auctions with independent private values.* This is almost a restatement of the midterm question, but in disguise. As in the midterm, incentive compatibility implies that for any bidder,

   $$U_i(v_i) = U_i(0) + \int_0^{v_i} Q_i(x)\, dx,$$

   where $U_i(v_i)$ is the expected value of the auction to bidder i, conditional on v_i, and Q_i is the probability that i wins, conditional on v_i. On the other hand,

   $$U_i(v_i) = Q_i(v_i)v_i - T_i(v_i)$$

   where T_i is the expected payment to the seller, conditional on v_i. Therefore,

   $$T_i(v_i) = Q_i(v_i)v_i - U_i(0) - \int_0^{v_i} Q_i(x)\, dx.$$

   The question asks about

   $$\mathbb{E}_{(v_1,\ldots,v_N)}\left[\sum_i T_i(v_i)\right].$$

   The critical point is that in any equilibrium of the stated type, the object goes to a bidder with highest value (because the object goes to the highest bidder, bids are increasing in value, and the bid functions are symmetric).

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[1] There are a number of ways to see this. One is to note that the subsets are formed by deciding for each strategy whether to include it in the subset. There are two possibilities for each strategy (included or not), and L strategies, so 2^L possible subsets of L. But this count includes the empty set. There are $2^L - 1$ non-empty subsets.

That is, the equilibrium is ex-post efficient. This implies that Q_i is the same in all equilibria under consideration. Since, by assumption, $U_i(0) = 0$ in all equilibria under consideration, $T_i(v_i)$ is the same under all equilibria under consideration, and therefore expected revenue is the same.

Note that you must prove that efficiency holds for equilibria of the stated type. Efficiency has not been assumed explicitly. Indeed, these auctions may have other equilibria, and for these equilibria the theorem may not hold. For example, the second price auction has an equilibrium in which bidder 1 bids 1, regardless of her value, and all other bidders bid 0, regardless of their values. This equilibrium yields 0 revenue to the seller.

Remarks. This is a version of the *Revenue Equivalence Theorem*, a basic result in auction theory. An early version of this goes back to Vickrey.

Here are three examples of auctions that have equilibria that fit the criteria of the result.

- First price auction (high bid wins, winner pays own bid, all others pay nothing).
- Third price auction (high bid wins and winner pays 3rd highest bid, all others pay nothing).
- All pay auction (high bid wins and each player pays own bid).

Even though these auctions have equilibria with the same expected revenue, these equilibria can generate very different revenues for any particular realization of bidder values. Thus, there are value profiles where the first price auction generates much higher revenue than the second price auction, and other value profiles where the second price auction generates much higher revenue than the first price auction. Revenue equivalence says that these differences exactly average out in expectation.

3. *The Patent War.*

   (a) If the state is B then 1 stays out (does not innovate) in any Nash equilibrium. In a separating equilibrium, 1 goes in when the state is G. This is optimal for 1 iff the payoff after innovating is non-negative. There are three sets of separating equilibria to look at, characterized by the following behavior in the subform following innovation.

      - 2 and 3 both play P, 1 plays C.
      - 2 and 3 both play C, 1 plays P.
      - 2 and 3 randomize 50:50 between P and C. 1 randomizes across P and C; since either yields an expected payoff of 1/4, any probability will do.

In each case, the payoff to 1 is positive, so it is in fact optimal for 1 to enter.

I also need to specify what happens when the state is B but 1 innovates (which does not occur along the path of play). Just have 1 withdraw and have 2 do the same thing as he did when the state is G (3 must do the same thing as when the state is G because 3 does not observe the state).

(b) I need to specify conditional probabilities at all information sets, show that these can be approximated using Bayes's rule via a sequence of fully mixed behavior strategy profiles converging to the equilibrium profile (that is, conditional probabilities satisfy "consistency"), and then show that, with these probabilities, all continuation strategies are sequentially rational. This is pretty much trivial, however. In a separating equilibrium of this game, all information sets following 'innovate' by player 1 in state G are along the path of play, and thus the conditional probabilities at nodes in those information sets (including all of player 3's nodes) are pinned down. For the game tree as indicated in part (c), the only other information set to consider is the one for player 1 following 'innovate' by player 1 in state B. Simply specify probabilities compatible with the behavior of player 2 and 3 in that subform (e.g., if player 2 and 3 play P, put probability 1 on the corresponding node for player 1). Note that the probabilities at player 1's nodes in this subform are irrelevant for his decision (he gets -1 no matter what he does).

(c) Suppose that 1 believes that 2 and 3 have either both played C or both played P, each with probability 1/2. Then either C or P for 1 earns -1/2. Thus player 1 is better off in this subform withdrawing and earning -.01. Given this, player 1 will choose not to innovate, and earn 0, in either state: player 1 will pool.

This pooling equilibrium is not sequential because this probability distribution across nodes in 1's information set cannot arise unless players 2 and 3 correlate, which consistency rules out.

More generally, for any independent randomization by 2 and 3, player 1 always earns at least 1/4 if 1 best responds (50:50 randomization by 2 and 3 minmaxes player 1 in the subform). So if conditional probabilities satisfy consistency then it is always sequentially rational for 1 to innovate when the state is G; there is no pooling sequential equilibrium.

4. *Long-Run Versus Short-Run.* In this game, the pro-consul is the long-run player (player 1) and the tribes are the short-run players (players 2).

(a) i. *Nash equilibrium.* The unique static Nash equilibrium is (B, F), with payoff 0 for player 1. (Notice that B is strictly dominant for player 1.) This means $n^1 = 0$. This is also player 1's minmax payoff.

ii. *Pure precommitment payoff.*

  - If player 1 chooses J, player 2's best response will be C, resulting in a payoff of 1 for player 1.
  - If player 1 chooses B, player 2's best response will be F, resulting in a payoff of 0 for player 1.

  Therefore player 1's pure precommitment action will be J, with payoff 1.

iii. *Mixed precommitment payoff.* Suppose player 1 plays J with probablity p and plays B with probability $1-p$. Then player 2's problem is to choose an action to maximize his payoff, with

$$U^2(F) = 0p + 1(1-p) = 1-p$$
$$U^2(C) = 1p + 0(1-p) = p$$

  Since player 1's payoff is maximized when player 2 plays C, player 1 chooses p so that, $U^2(C) \geq U^2(F)$, hence $p \geq \frac{1}{2}$.
  Given $p \geq \frac{1}{2}$, player 1's payoff is $1p + 2(1-p) = 2-p$, which is decreasing in p. Hence player 1 wants to set p as low as possible, namely $p = 1/2$. In summary, player 1's best precommitment strategy is to randomize 50:50 between J and B, with expected payoff $3/2$.

(b) Denote the best dynamic equilibrium payoff by $\overline{v}^1$. Suppose player 1 plays J with probablity p and plays B with probability $1-p$.

The maximum equilibrium payoff average for player 1 is given by

$$\overline{v}^1 = \max_{\alpha | \alpha^2 \in \mathrm{BR}^2(\alpha^1)} \; \min_{a^1 | \alpha^1(a^1) > 0} u^1\left(a^1, \alpha^2\right),$$

which can be calculated as follows

p	$\mathrm{BR}^2(\alpha^1)$	min	max min
$p = 0$	F	0	
$0 < p < \frac{1}{2}$	F	-1	
$p = \frac{1}{2}$	C, F	-1	
$\frac{1}{2} < \mathbf{p} < 1$	$\mathbf{C}$	**1**	**1**
$\mathbf{p = 1}$	$\mathbf{C}$	**1**	**1**

Therefore, $\overline{v}^1 = 1$. This can be achieved by having player 1 play J and player 2 play C along the path of play, with reversion to the stage game Nash equilibrium if player 1 ever deviates (recall that the stage game NE payoff to the pro-consul is also his minmax payoff). Deviation by player 1, to B, will not be profitable if

$$\overline{v}^1 \geq (1-\delta)\, 2 + \delta n^1$$

Since $\overline{v}^1 = 1$ and $n^1 = 0$, this requires $\delta \geq 1/2$. In summary, there is an equilibrium in which the pro-consul gets more than $n^1 = 0$ if and only if $\delta \geq 1/2$, and in this case the pro-consul can get a maximum of $\overline{v}^1 = 1$.

(c) If 2 cannot observe player 1's action in previous periods, the equilibrium payoff for the next period on is dependent on the signal $\{j, b\}$ instead of player 1's action $\{J, B\}$.

Denote the best dynamic equilibrium payoff by $\overline{v}^1$. $\overline{v}^1$ must obey the following conditions.

$$\begin{aligned}
\overline{v}^1 &= (1-\delta)\,1 + \delta\left[0.9w^1(j) + 0.1w^1(b)\right] \\
\overline{v}^1 &\geq (1-\delta)\,2 + \delta\left[0.1w^1(j) + 0.9w^1(b)\right] \\
0 &\leq w^1(j) \leq \overline{v}^1 \\
0 &\leq w^1(b) \leq \overline{v}^1
\end{aligned}$$

Maximization of $\overline{v}^1$ requires that

$$w^1(j) = \overline{v}^1$$

and also that the second inequality (the no profitable deviation inequality) hold with equality,

$$\overline{v}^1 = (1-\delta)\,2 + \delta\left[0.1w^1(j) + 0.9w^1(b)\right].$$

Solving for $\overline{v}^1$ and $w^1(j)$ yields

$$\begin{aligned}
\overline{v}^1 &= \frac{7}{8}, \\
w^1(b) &= \frac{17}{8} - \frac{10}{8\delta}.
\end{aligned}$$

Since $0 \leq w^1(b) \leq \overline{v}^1$, it follows that

$$0 \leq \frac{17}{8} - \frac{10}{8\delta} \leq \frac{7}{8},$$

The second of these inequalities holds for any $\delta \leq 1$. The first of these inequalities holds for any

$$\delta \geq \frac{10}{17}.$$

In summary, there is an equilibrium in which the pro-consul gets more than $n^1 = 0$ if and only if $\delta \geq 10/17$ (which is greater than $1/2$) and in this case the pro-consul can get a maximum of $7/8$ (which is less than 1).